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Miscellanea

Nearly orthogonal arrays mappable into fully orthogonal arrays

BY RAHUL MUKERJEE

Indian Institute of Management Calcutta, Joka, Diamond Harbour Road, Kolkata 700 104, India
rmuk0902@gmail.com

FASHENG SUN

School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China
sfxsfx2001@gmail.com

AND BOXIN TANG

Department of Statistics and Actuarial Science, Simon Fraser University, Burnaby, British Columbia V5A 1S6, Canada
boxint@sfu.ca

SUMMARY

We develop a method for construction of arrays which are nearly orthogonal, in the sense that each column is orthogonal to a large proportion of the other columns, and which are convertible to fully orthogonal arrays via a mapping of the symbols in each column to a possibly smaller set of symbols. These arrays can be useful in computer experiments as designs which accommodate a large number of factors and enjoy attractive space-filling properties. Our construction allows both the mappable nearly orthogonal array and the consequent fully orthogonal array to be either symmetric or asymmetric. Resolvable orthogonal arrays play a key role in the construction.

Some key words: Computer experiment; Difference scheme; Resolvable array; Space-filling design.

1. INTRODUCTION

Ever since the seminal work of Rao (1947), orthogonal arrays have become so prominent in the design literature as to form the backbone of designs for multifactor experiments. An $N \times m$ array with symbols $0, 1, \ldots, s-1$ is an orthogonal array of strength two if in every two columns the s^2 ordered pairs of symbols occur with the same frequency. We write $\text{OA}(N; s^m)$ for such an array; a more general definition appears in § 2. When used in computer experiments (Santner et al., 2003), orthogonal arrays provide an easy construction of an attractive class of space-filling designs (Owen, 1992; Tang, 1993). Dey & Mukerjee (1999) reviewed the construction and optimality of orthogonal arrays as fractional factorial designs. For a comprehensive treatment of these arrays, see Hedayat et al. (1999).

Motivated by computer experiments, in this paper we introduce and construct a class of arrays, called mappable nearly orthogonal arrays. Such an array is nearly orthogonal in the sense that each column is orthogonal to a large proportion of the other columns, and the array becomes a fully orthogonal array when its symbols are collapsed into a smaller set of symbols. To see the benefits of using such an array in computer experiments, we provide an illustrative example. An $\text{OA}(81; 9^{10})$ allows the construction of a design of 81 runs for 10 factors that achieves a stratification on a 9×9 grid in all two-dimensions. If an $\text{OA}(81; 3^{40})$

is used, one can also construct a design of 81 runs that can accommodate 40 factors but achieves a stratification only on a coarser 3×3 grid in all two-dimensions. A mappable nearly orthogonal array constructible from the theory of this paper has 81 runs and can accommodate 40 factors each of 9 symbols. This design is able to achieve a stratification on a 9×9 grid in 720 out of all 780 two-dimensions and on a 3×3 grid in the remaining 60 two-dimensions. Thus it enjoys much better space-filling properties than an $\mathrm{OA}(81; 3^{40})$, while at the same time accommodating a considerably larger number of factors than an $\mathrm{OA}(81; 9^{10})$.

Popular choices of models for computer experiments are those based on Gaussian processes, and, correspondingly, space-filling designs have been widely accepted as appropriate designs (Santner et al., 2003). One can evaluate the uniformity of a design using distance criteria (Johnson et al., 1990) or discrepancy criteria (Fang & Mukerjee, 2000). However, for a high-dimensional input space, it is more fruitful to consider designs that are space-filling in lower-dimensional projections. Latin hypercube designs (McKay et al., 1979), designs based on orthogonal arrays (Owen, 1992; Tang, 1993) and strong orthogonal arrays (He & Tang, 2013) are a sequence of steps that aim at low-dimensional space-filling. The present work, in our opinion, represents another direction in the quest for even better designs that are space-filling in lower dimensions.

The notion of near orthogonality conceived in this paper is very different from the traditional formulation in the literature on nearly orthogonal arrays. The traditional formulation is combinatorial in nature and does not involve symbol collapsing; see Xu (2002) for a detailed discussion and further references. Incidentally, the idea of symbol collapsing appears also in Qian & Wu (2009). However, their objectives and results are different from ours; for instance, their arrays are fully orthogonal even before symbol collapsing and thus accommodate fewer factors than those explored here.

2. DEFINITIONS AND PRELIMINARIES

Two $N \times 1$ column vectors a_1 and a_2, populated by s_1 and s_2 symbols, respectively, are said to be orthogonal if all $s_1 s_2$ ordered pairs of symbols occur equally often as rows in the $N \times 2$ array $(a_1\ a_2)$. An orthogonal array $\mathrm{OA}(N; \prod_{j=1}^m s_j)$ of strength two is an $N \times m$ array, with its m columns populated by $s_1, \ldots, s_m$ symbols, such that every two distinct columns are orthogonal. In the symmetric case where $s_j = s$ for each j, an $\mathrm{OA}(N; \prod_{j=1}^m s_j)$ is denoted simply by $\mathrm{OA}(N; s^m)$. Similar simplified notation will be used if the s_j are equal in clusters.

DEFINITION 1. *A mappable nearly orthogonal array* $\mathrm{MNOA}(R; \prod_{j=1}^m s_j^{u_j}, \prod_{j=1}^m \prod_{k=1}^{u_j} p_{jk})$ *is an* $R \times \tilde{u}$ *array whose* $\tilde{u} = u_1 + \cdots + u_m$ *columns can be partitioned into* m *disjoint groups of* $u_1, \ldots, u_m$ *columns with the following properties:*

(i) *for* $j = 1, \ldots, m$, *every column of the* j*th group is populated by* s_j *symbols;*
(ii) *any two columns from different groups are orthogonal;*
(iii) *for* $j = 1, \ldots, m$ *and* $k = 1, \ldots, u_j$, *the* s_j *symbols in the* k*th column of the* j*th group can be mapped to a set of* $p_{jk} \leqslant s_j$ *symbols such that these mappings convert the array into an orthogonal array* $\mathrm{OA}(R; \prod_{j=1}^m \prod_{k=1}^{u_j} p_{jk})$.

In particular, if $s_j = s$, $u_j = u$ and $p_{jk} = p$ for every j and k, then a mappable nearly orthogonal array as in Definition 1 will be denoted by $\mathrm{MNOA}\{R; (s^u)^m, (p^u)^m\}$. Similar simplified and self-evident notation will be used also when the s_j, u_j or p_{jk} are equal in clusters.

By Definition 1(ii), in a mappable nearly orthogonal array before mapping, each of the u_j columns in the jth group is orthogonal to at least a proportion $\pi_j = (\tilde{u} - u_j)/(\tilde{u} - 1)$ of the other columns. This leads to the following measures of the pre-mapping degree of orthogonality among the columns:

$$\bar{\pi} = \sum_{j=1}^m u_j \pi_j \bigg/ \sum_{j=1}^m u_j = \left(\tilde{u}^2 - \sum_{j=1}^m u_j^2\right) \bigg/ \{\tilde{u}(\tilde{u} - 1)\} \tag{1}$$

and

$$\pi_{\min} = \min_{1 \leqslant j \leqslant m} \pi_j = \left(\tilde{u} - \max_{1 \leqslant j \leqslant m} u_j\right) \bigg/ (\tilde{u} - 1). \tag{2}$$

Table 1. *Pre-mapping and post-mapping* MNOA$\{16;\ (4^3)^5,\ (2^3)^5\}$

Pre-mapping

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	2	2	1	2	2	1	2	2	1	2	2
0	0	0	2	1	3	2	1	3	2	1	3	2	1	3
0	0	0	3	3	1	3	3	1	3	3	1	3	3	1
1	2	2	0	0	0	1	2	2	2	1	3	3	3	1
1	2	2	1	2	2	0	0	0	3	3	1	2	1	3
1	2	2	2	1	3	3	3	1	0	0	0	1	2	2
1	2	2	3	3	1	2	1	3	1	2	2	0	0	0
2	1	3	0	0	0	2	1	3	3	3	1	1	2	2
2	1	3	1	2	2	3	3	1	2	1	3	0	0	0
2	1	3	2	1	3	0	0	0	1	2	2	3	3	1
2	1	3	3	3	1	1	2	2	0	0	0	2	1	3
3	3	1	0	0	0	3	3	1	1	2	2	2	1	3
3	3	1	1	2	2	2	1	3	0	0	0	3	3	1
3	3	1	2	1	3	1	2	2	3	3	1	0	0	0
3	3	1	3	3	1	0	0	0	2	1	3	1	2	2

Post-mapping

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	1	0	1	1	0	1	1	0	1	1
0	0	0	1	0	1	1	0	1	1	0	1	1	0	1
0	0	0	1	1	0	1	1	0	1	1	0	1	1	0
0	1	1	0	0	0	0	1	1	1	0	1	1	1	0
0	1	1	0	1	1	0	0	0	1	1	0	1	0	1
0	1	1	1	0	1	1	1	0	0	0	0	0	1	1
0	1	1	1	1	0	1	0	1	0	1	1	0	0	0
1	0	1	0	0	0	1	0	1	1	1	0	0	1	1
1	0	1	0	1	1	1	1	0	1	0	1	0	0	0
1	0	1	1	0	1	0	0	0	0	1	1	1	1	0
1	0	1	1	1	0	0	1	1	0	0	0	1	0	1
1	1	0	0	0	0	1	1	0	0	1	1	1	0	1
1	1	0	0	1	1	1	0	1	0	0	0	1	1	0
1	1	0	1	0	1	0	1	1	1	1	0	0	0	0
1	1	0	1	1	0	0	0	0	1	0	1	0	1	1

In particular, if $u_1 = \cdots = u_m = u$, then $\tilde{u} = mu$ and by (1) and (2) we have

$$\bar{\pi} = \pi_{\min} = (m-1)u/(mu-1). \tag{3}$$

Example 1. In Table 1, the array on the left is a pre-mapping MNOA$\{16;\ (4^3)^5,\ (2^3)^5\}$, where each of the 15 columns is populated by four symbols. If we partition these columns into five disjoint groups of three columns each such that the first group consists of the first three columns, the second group consists of the next three columns, and so on, then any two distinct columns are orthogonal if and only if they are from different groups. Thus, with $m = 5$ and $u = 3$ in (3), we obtain the pre-mapping degree of orthogonality as $\bar{\pi} = \pi_{\min} = 12/14 = 6/7$. Now suppose that the four symbols in each column are mapped to two symbols by $0, 1 \rightarrow 0$ and $2, 3 \rightarrow 1$. Then the resulting array is seen to be a two-symbol orthogonal array, which is the array displayed on the right in Table 1. In this example, the same mapping works for all columns, because they all have four symbols and are mapped to two-symbol columns. As will be seen later, this is not the case in general.

3. Method of construction

Resolvable orthogonal arrays will be very useful in constructing mappable nearly orthogonal arrays. An OA$(N; \prod_{j=1}^m s_j)$, say C, is resolvable into λ parts if it can be partitioned as $C = (C_1^{\mathrm{T}}, \ldots, C_\lambda^{\mathrm{T}})^{\mathrm{T}}$ such that each subarray C_w is $(N/\lambda) \times m$ and forms an orthogonal array of strength one, i.e., in each C_w the s_j symbols occur equally often in the jth column, for $j = 1, \ldots, m$. Here the superscript T stands for transpose. We write OA$_\lambda(N; \prod_{j=1}^m s_j)$ for such a resolvable orthogonal array. An OA$_\lambda(N; \prod_{j=1}^m s_j)$ is co-existent with an OA$(N; \lambda \times \prod_{j=1}^m s_j)$, which can be easily seen as follows. The former can be obtained from the latter simply by forming the λ subarrays according to the λ symbols in the first column of the latter and then deleting this first column. Conversely, from an OA$_\lambda(N; \prod_{j=1}^m s_j)$, $C = (C_1^{\mathrm{T}}, \ldots, C_\lambda^{\mathrm{T}})^{\mathrm{T}}$, one can construct an OA$(N; \lambda \times \prod_{j=1}^m s_j)$ by adding one column to C in such a way that the entries of this new column that correspond to C_w are equal to $w - 1$ for $w = 1, \ldots, \lambda$. Incidentally, our construction, described in Theorem 1 below, also covers the trivial case of $\lambda = 1$, where no partitioning into subarrays takes place at all.

THEOREM 1. *Suppose that there exist an* OA$(N; \prod_{j=1}^m s_j)$*, say B, and resolvable arrays* OA$_\lambda(\lambda s_j; \prod_{k=1}^{u_j} p_{jk})$*, say $C^{(j)}$ $(j = 1, \ldots, m)$. Then an* MNOA$(\lambda N; \prod_{j=1}^m s_j^{u_j}, \prod_{j=1}^m \prod_{k=1}^{u_j} p_{jk})$ *can be constructed.*

Proof. We first describe the construction via several steps and then show its validity.

Step 1. For $j = 1, \ldots, m$ and $k = 1, \ldots, u_j$, let $C^{(j)}$ have symbols $0, 1, \ldots, p_{jk} - 1$ in the kth column. Write $C_w^{(j)}$ ($w = 1, \ldots, \lambda$) for the subarrays of $C^{(j)}$ such that each $C_w^{(j)}$ is $s_j \times u_j$ and forms an orthogonal array of strength one. Then $s_j = t_{jk} p_{jk}$ for some integer t_{jk}. For each fixed j and k, consider a partitioning of the set $\{0, 1, \ldots, s_j - 1\}$ into p_{jk} mutually exclusive and exhaustive subsets each of cardinality t_{jk}, as given by

$$S_{jkh} = \{ht_{jk}, ht_{jk} + 1, \ldots, ht_{jk} + t_{jk} - 1\}, \quad h = 0, \ldots, p_{jk} - 1. \tag{4}$$

Step 2. For each j, w, k and h, in the kth column of $C_w^{(j)}$ replace the $s_j / p_{jk} = t_{jk}$ occurrences of the symbol h by the t_{jk} members of S_{jkh} as per the order in (4); that is, the first occurrence of h is replaced by ht_{jk}, the second occurrence by $ht_{jk} + 1$, and so on. Let $D_w^{(j)}$ be the $s_j \times u_j$ array obtained from $C_w^{(j)}$ in this manner. Each column of $D_w^{(j)}$ is a permutation of $0, 1, \ldots, s_j - 1$. Let $d_w^{(j)}(0), d_w^{(j)}(1), \ldots, d_w^{(j)}(s_j - 1)$ denote the s_j rows of $D_w^{(j)}$.

Step 3. Turning now to the orthogonal array B, write $B = (b_{ij})$ and let $0, 1, \ldots, s_j - 1$ be the symbols in its jth column, so that $b_{ij} \in \{0, 1, \ldots, s_j - 1\}$ for $i = 1, \ldots, N$ and $j = 1, \ldots, m$. Construct the following arrays:

(a) $A_w^{(j)}$, of order $N \times u_j$, having rows $d_w^{(j)}(b_{1j}), d_w^{(j)}(b_{2j}), \ldots, d_w^{(j)}(b_{Nj})$, for $j = 1, \ldots, m$ and $w = 1, \ldots, \lambda$;
(b) $A^{(j)} = (A_1^{(j)\mathrm{T}}, \ldots, A_\lambda^{(j)\mathrm{T}})^\mathrm{T}$, of order $\lambda N \times u_j$, for $j = 1, \ldots, m$;
(c) $A = (A^{(1)}, \ldots, A^{(m)})$, of order $\lambda N \times \tilde{u}$, where $\tilde{u} = u_1 + \cdots + u_m$.

We now show that the array A constructed as above is the desired mappable nearly orthogonal array. Partition the columns of A into m disjoint groups, such that the jth group consists of the u_j columns of $A^{(j)}$ ($j = 1, \ldots, m$). In conformity with Definition 1(i), by Step 2 and Step 3(a) and (b), each column in the jth group is populated by s_j symbols $0, 1, \ldots, s_j - 1$. Next, for each j and w, by Step 2 and Step 3(a), every column of $A_w^{(j)}$ is obtained by permuting the symbols of the jth column of B. Since B is an orthogonal array, it follows that for every w and $j_1 \neq j_2$, each column of $A_w^{(j_1)}$ is orthogonal to each column of $A_w^{(j_2)}$, i.e., by Step 3(b), any two columns of A from different groups are orthogonal, as stipulated in Definition 1(ii). Finally, for $j = 1, \ldots, m$ and $k = 1, \ldots, u_j$, consider the kth column of the jth group and map the s_j symbols in this column to p_{jk} symbols as dictated by (4), i.e.,

$$ht_{jk}, ht_{jk} + 1, \ldots, ht_{jk} + t_{jk} - 1 \to h, \quad h = 0, \ldots, p_{jk} - 1.$$

By Step 2, these mappings revert $D_w^{(j)}$ to $C_w^{(j)}$ and hence, by Step 3(a), convert $A_w^{(j)}$ to an $N \times u_j$ array whose rows are given by those of $C_w^{(j)}$, each repeated N/s_j times. By Step 3(b), this in turn converts $A^{(j)}$ to a $\lambda N \times u_j$ array whose rows are given by those of $C^{(j)}$, each repeated N/s_j times. Since each $C^{(j)}$ is an orthogonal array, it follows that, post-mapping, any two distinct columns of $A^{(j)}$ are orthogonal. Furthermore, as already noted, any two columns of A from different groups, $A^{(j_1)}$ and $A^{(j_2)}$ for $j_1 \neq j_2$, are orthogonal pre-mapping and hence remain orthogonal post-mapping. Therefore, in conformity with Definition 1(iii), post-mapping, A is an $\mathrm{OA}(\lambda N; \prod_{j=1}^{m} \prod_{k=1}^{u_j} p_{jk})$. □

Remark 1. As with ordinary orthogonal arrays, if $\lambda > 1$ and the array A constructed in Theorem 1 is augmented by a column

$$\xi = (0, \ldots, 0, 1, \ldots, 1, \ldots, \lambda - 1, \ldots, \lambda - 1)^\mathrm{T}$$

where each symbol $0, 1, \ldots, \lambda - 1$ is repeated N times, then this additional λ-symbol column is orthogonal to each column of A pre-mapping and, therefore, post-mapping.

Remark 2. For $\lambda = 1$, if B and $C^{(j)}$ ($j = 1, \ldots, m$) are saturated orthogonal arrays, then, post-mapping, A is a saturated orthogonal array. For $\lambda > 1$, if B and the arrays $\mathrm{OA}(\lambda s_j; \lambda \times \prod_{k=1}^{u_j} p_{jk})$, which are coexistent with $C^{(j)}$ ($j = 1, \ldots, m$), are saturated, then, post-mapping, the array A when augmented by ξ is a saturated orthogonal array.

Various existence and construction results for the arrays in Theorem 1(i) and (ii) are available in Hedayat et al. (1999). Some of these results will be used in the next section when we discuss applications of

Table 2. *Some mappable nearly orthogonal arrays of Series* 1–4

Series	p	α	Array	$\bar{\pi}$	$\pi_{\min}$
1	2	2	$\text{MNOA}\{16; (4^3)^5, (2^3)^5\}$	0·8571	0·8571
1	2	3	$\text{MNOA}\{64; (8^7)^9, (2^7)^9\}$	0·9032	0·9032
1	2	4	$\text{MNOA}\{256; (16^{15})^{17}, (2^{15})^{17}\}$	0·9449	0·9449
1	3	2	$\text{MNOA}\{81; (9^4)^{10}, (3^4)^{10}\}$	0·9231	0·9231
1	4	2	$\text{MNOA}\{256; (16^5)^{17}, (4^5)^{17}\}$	0·9524	0·9524
2	2	2	$\text{MNOA}\{32; (4^6)^5, (2^6)^5\}$	0·8276	0·8276
2	2	3	$\text{MNOA}\{128; (8^{14})^9, (2^{14})^9\}$	0·8960	0·8960
2	2	4	$\text{MNOA}\{512; (16^{30})^{17}, (2^{30})^{17}\}$	0·9430	0·9430
2	3	2	$\text{MNOA}\{243; (9^{12})^{10}, (3^{12})^{10}\}$	0·9076	0·9076
2	4	2	$\text{MNOA}\{1024; (16^{20})^{17}, (4^{20})^{17}\}$	0·9440	0·9440
3	2	3	$\text{MNOA}\{128; (8^6)^9, (2^2 \times 4^4)^9\}$	0·9057	0·9057
3	2	4	$\text{MNOA}\{1024; (16^{12})^{17}, (2^4 \times 8^8)^{17}\}$	0·9458	0·9458
4	2	3	$\text{MNOA}\{64; (8^6) \times (4^4)^8, (2^2 \times 4^4) \times (4 \times 2^3)^8\}$	0·9104	0·8649
4	2	4	$\text{MNOA}\{512; (16^{12}) \times (8^8)^{16}, (2^4 \times 8^8) \times (8 \times 4^7)^{16}\}$	0·9472	0·9209
4	3	3	$\text{MNOA}\{729; (27^{12}) \times (9^9)^{27}, (3^3 \times 9^9) \times (9 \times 3^8)^{27}\}$	0·9679	0·9567

Theorem 1. We conclude this section by noting that our method of constructing mappable nearly orthogonal arrays is different from, though similar in spirit to, the substitution method of Liu & Cai (2009) for constructing mixed-level supersaturated designs.

4. Applications

For illustration, we now use Theorem 1 to obtain four series of mappable nearly orthogonal arrays. Table 2 summarizes some of the arrays in these series. Many other series can be found similarly from Theorem 1.

Series 1. Let p be a prime or prime power and let $s = p^\alpha$, where $\alpha \geqslant 2$ is an integer. In Theorem 1, if one takes B as an $\text{OA}(s^2; s^{s+1})$ and, with $\lambda = 1$, each $C^{(j)}$ as an $\text{OA}(p^\alpha; p^{f_1})$ where $f_1 = (p^\alpha - 1)/(p-1)$, then one obtains a symmetric $\text{MNOA}\{s^2; (s^{f_1})^{s+1}, (p^{f_1})^{s+1}\}$. The case of $p = \alpha = 2$ yields the mappable nearly orthogonal array in Example 1. By (3), the pre-mapping degree of orthogonality of this array is given by $\bar{\pi} = \pi_{\min} = sf_1/\{(s+1)f_1 - 1\}$.

Series 2. Let p, s and α be as in Series 1. In Theorem 1, if one takes B as an $\text{OA}(s^2; s^{s+1})$ and, with $\lambda = p$, each $C^{(j)}$ as an $\text{OA}_p(p^{\alpha+1}; p^{f_2})$ where $f_2 = (p^{\alpha+1} - 1)/(p-1) - 1$, then one obtains a symmetric $\text{MNOA}\{ps^2; (s^{f_2})^{s+1}, (p^{f_2})^{s+1}\}$. By (3), for this array, $\bar{\pi} = \pi_{\min} = sf_2/\{(s+1)f_2 - 1\}$. The resolvable arrays $C^{(j)}$ considered here can easily be obtained from a saturated $\text{OA}(p^{\alpha+1}; p^{f_2+1})$ as indicated at the beginning of § 3.

Series 3. Let p and $s = p^\alpha$ be as in Series 1, now with $\alpha \geqslant 3$. Write $c = p^{\alpha-1}$ and $g = p^{\alpha-2}$. Using a difference scheme as discussed in Theorem 6.6 of Hedayat et al. (1999), one can then find an $\text{OA}(c; g \times p^g)$, say L_1, with c rows and $g + 1$ columns of which the first is populated by g symbols and the rest with p symbols each. Next, consider a symmetric $\text{OA}(c^2; c^{c+1})$, and in its first column replace the c symbols by the c rows of L_1 to get an $\text{OA}(c^2; g \times p^g \times c^c)$. This in turn yields a resolvable $\text{OA}_\lambda(c^2; p^g \times c^c)$, say L_2, where $\lambda = g$. Now, in Theorem 1, take B to be an $\text{OA}(s^2; s^{s+1})$ and each $C^{(j)}$ to be the resolvable array L_2. This is feasible as $c^2 = \lambda s$, and yields an $\text{MNOA}\{gs^2; (s^{g+c})^{s+1}, (p^g \times c^c)^{s+1}\}$ which has $(g+c)(s+1)$ columns such that: (a) pre-mapping, each column is populated by s symbols and the columns are partitioned into $s + 1$ groups of $g + c$ columns each, as envisaged in Definition 1; (b) post-mapping, of the $g + c$ columns in each group, g convert to p-symbol columns and c convert to c-symbol columns, so that the array becomes an orthogonal array of strength two. By (3), for this array, $\bar{\pi} = \pi_{\min} = s(g+c)/\{(s+1)(g+c) - 1\}$.

Series 4. Let p, s, c, g, α and the resolvable $\text{OA}_\lambda(c^2; p^g \times c^c)$, called L_2, be as in Series 3, where $\lambda = g$. As before, using difference schemes, one can also find an $\text{OA}(sc; s \times c^s)$, say L_0, and an $\text{OA}(cg; c \times g^c)$ that yields a resolvable $\text{OA}_\lambda(cg; c \times g^{c-1})$, say L_3, where $\lambda = g$. In Theorem 1, take B to be L_0, $C^{(1)}$ to be the resolvable array L_2, and each of the other $C^{(j)}$ to be the resolvable array L_3. This is feasible as

Table 3. *Some mappable nearly orthogonal arrays not covered by Series* 1–4

B	λ	$C^{(j)}$ for every j	Mappable nearly orthogonal array	$\bar{\pi} = \pi_{\min}$
OA$(16; 4^5)$	6	OA$_6(24; 2^{14})$	MNOA$\{96; (4^{14})^5, (2^{14})^5\}$	0·8116
OA$(81; 9^{10})$	2	OA$_2(18; 3^7)$	MNOA$\{162; (9^7)^{10}, (3^7)^{10}\}$	0·9130
OA$(64; 8^9)$	3	OA$_3(24; 2^{16})$	MNOA$\{192; (8^{16})^9, (2^{16})^9\}$	0·8951
OA$(81; 9^{10})$	4	OA$_4(36; 3^{13})$	MNOA$\{324; (9^{13})^{10}, (3^{13})^{10}\}$	0·9070
OA$(64; 8^9)$	6	OA$_6(48; 2^2 \times 4^{12})$	MNOA$\{384; (8^{14})^9, (2^2 \times 4^{12})^9\}$	0·8960
OA$(144; 12^7)$	3	OA$_3(36; 3^6 \times 6^3)$	MNOA$\{432; (12^9)^7, (3^6 \times 6^3)^7\}$	0·8710

$c^2 = \lambda s$ and $cg = \lambda c$, and yields an MNOA$\{gsc;\ (s^{g+c}) \times (c^c)^s, (p^g \times c^c) \times (c \times g^{c-1})^s\}$ which involves $s + 1$ groups of columns, with $g + c$ columns in the first group and c columns in each of the other groups. Hence, by (1) and (2),

$$\bar{\pi} = \frac{sc(2g + c + sc)}{(g + c + sc)(g + c + sc - 1)}, \quad \pi_{\min} = \frac{sc}{g + c + sc - 1}.$$

Here, one column from each of the last s groups has c symbols both pre- and post-mapping. However, one can check that it is orthogonal to the other columns of the same group only post-mapping.

Remark 3. As observed in Remark 1, mappable nearly orthogonal arrays of Series 2, 3 and 4 can be augmented by a λ-symbol column, with $\lambda = p$, g and g, respectively, retaining orthogonality both pre- and post-mapping. Also, following Remark 2, one can check that, post-mapping, mappable nearly orthogonal arrays of Series 1 are saturated. Furthermore, the same holds for mappable nearly orthogonal arrays of Series 2–4 upon augmentation by a λ-symbol column as indicated above.

For mappable nearly orthogonal arrays of Series 1–4, the numbers of rows and the numbers of symbols in all columns, both pre- and post-mapping, are primes or prime powers. There are many other applications of Theorem 1 where this is not the case. Some such applications are indicated in Table 3. These arrays are, however, not saturated post-mapping, even upon augmentation by a λ-symbol column, because the relevant conditions in Remark 2 do not hold.

Remark 4. In any specific situation, if necessary, one or more columns of a mappable nearly orthogonal array can be deleted to get another mappable nearly orthogonal array with the same number of rows but fewer columns. The expressions for $\bar{\pi}$ and $\pi_{\min}$ in (1) and (2) suggest that while doing so, one should attempt to keep the group sizes as close to equal as possible without reducing the number of groups. For instance, in order to obtain a mappable nearly orthogonal array with 81 rows and 28 columns each having 9 symbols pre-mapping, one can start with the MNOA$\{81; (9^4)^{10}, (3^4)^{10}\}$ in Table 2, and delete two columns from each of the first two groups and one column from each of the last eight groups. By (1) and (2), the resulting MNOA$\{81; (9^2)^2 \times (9^3)^8, (3^2)^2 \times (3^3)^8\}$ has $\bar{\pi} = 0{\cdot}9312$ and $\pi_{\min} = 0{\cdot}9259$. On the other hand, starting from the same MNOA$\{81; (9^4)^{10}, (3^4)^{10}\}$, if one deletes the last three groups of columns altogether, then the resulting MNOA$\{81; (9^4)^7, (3^4)^7\}$, also with 28 columns, has lower values of $\bar{\pi}$ and $\pi_{\min}$, namely $\bar{\pi} = \pi_{\min} = 0{\cdot}8889$, because of a reduction in the number of groups.

5. Concluding remarks

The research presented in this paper opens up several possible directions for future work. One natural direction is to construct the higher-strength versions of mappable nearly orthogonal arrays. In a recent paper, He & Tang (2013) introduced strong orthogonal arrays for computer experiments. It would be interesting to explore whether or not a marriage between these and mappable nearly orthogonal arrays could produce even better designs, if such a marriage is indeed possible.

The arrays constructed in this paper are intended to achieve better low-dimensional space-filling properties as compared with their post-mapping counterparts. As has been mentioned in § 1, one may want to further evaluate mappable nearly orthogonal arrays using distance or discrepancy criteria. Starting with a given mappable nearly orthogonal array, one can generate a class of arrays by permuting its levels in each column, and the resulting arrays will still be mappable nearly orthogonal arrays provided level permutations do not damage the property of being an orthogonal array post-mapping. For example, in the MNOA$\{16; (4^3)^5, (2^3)^5\}$ in Table 1, if we replace the four levels 0, 1, 2 and 3 in the first column by 1, 0, 3 and 2, respectively, the resulting array is still an MNOA$\{16; (4^3)^5, (2^3)^5\}$. A practically important problem would be to select better arrays from the class of all such mappable nearly orthogonal arrays using a distance or discrepancy criterion. This is in the same spirit as finding maximin Latin hypercubes (Morris & Mitchell, 1995) and optimal orthogonal-array-based Latin hypercubes (Leary et al., 2003).

Pre-mapping, mappable nearly orthogonal arrays can be viewed directly as multilevel supersaturated designs. Constructions and the optimality of such designs have been considered by many authors, including Xu & Wu (2005) and Liu & Cai (2009). One commonly used optimality criterion for multilevel supersaturated designs is that of $E(f_{\text{NOD}})$ (Fang et al., 2003). The study of mappable nearly orthogonal arrays from this perspective will at least be of theoretical importance. Designs for computer experiments with two levels of accuracy have been gaining in popularity recently (Qian et al., 2009). It would be interesting to study whether and how the ideas of mappable nearly orthogonal arrays might be useful for this purpose.

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